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2005 J. Phys. A: Math. Gen. 38 L707

(http://iopscience.iop.org/0305-4470/38/43/L02)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) L707–L714 doi:10.1088/0305-4470/38/43/L02

LETTER TO THE EDITOR

Generalized Sommerfeld integral and diffraction in an angle-shaped domain with a radial perturbation

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Received 8 July 2005, in final form 12 September 2005
Published 12 October 2005
Online at stacks.iop.org/JPhysA/38/L707

Abstract
The Sommerfeld integral enabled one to solve various diffraction problems in angle-shaped domains. The basic goal of the present paper is to demonstrate a generalized form of the famous Sommerfeld integral. The kernel of the traditional Sommerfeld integral is replaced by the solution of a scattering problem. The latter hint leads to the possibility of representing solutions for a more general class of diffraction problems in an angle-shaped domain with the radial perturbation of the boundary or (and) with the presence of a radially symmetric refraction index in the stationary wave (Helmholtz) equation. The approach is demonstrated in the simplest situation of diffraction in an angle-shaped domain with the centrally circular (with respect to the angle vertex) perturbation of the perfectly conducting boundary.

PACS numbers: 02.60.Nm, 42.25.Fx

In the papers (Tai 1994, Luk'yanov and Nikitin 2000, Lavrov and Luk'yanov 2002) the authors demonstrated that the explicit solution of the diffraction problems can be obtained in some angular domains with the boundary supplemented by the circular part. We introduce a generalized form of the well-known Sommerfeld integral and exploit it for construction of the solution to a diffraction problem similar to those mentioned above. The corresponding problems can be considered as important new canonical problems solved by means of the generalized Sommerfeld integral. These problems may also be used in frames of the geometrical optics conceptions for different research and engineering applications.

1. The formulation

Consider the domain $\Omega = \{(r, \varphi) : r > a, |\varphi| < \Phi\}$, figure 1(*A*). The plane wave

$$u^{\mathrm{i}}(kr, \varphi) = \exp(-\mathrm{i}kr\cos(\varphi - \varphi_0)) \tag{1}$$

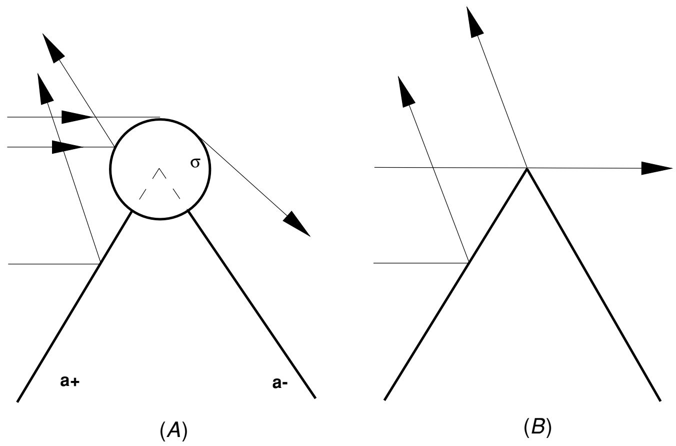

Figure 1. Diffraction by an angle with a radial perturbation (A); diffraction by a non-perturbed angle (B).

is incident on the boundary $\partial\Omega = \sigma \cup a_+ \cup a_-$ consisting of the circumference arc σ ($r = a, |\varphi| < \Phi$) of the radius a with the centre at the vertex O of the angle, $a_\pm$ are the parts of angle sides for $r > a, \varphi = \pm\Phi, 2\Phi (> \pi)$ is the opening of the angle, (r, φ) are the polar coordinates, $k > 0$ is the wave number. The parameter φ_0 is the (incidence) angle between the OX-axis and the direction from which the plane incident wave comes, $-\mathrm{i}kr\cos(\varphi - \varphi_0) = -\mathrm{i}k(x\cos\varphi_0 + y\sin\varphi_0)$, $x = r\cos\varphi$, $y = r\sin\varphi$.

The total wave field

$$u(r,\varphi) = u^{\mathrm{i}}(r,\varphi) + u^{\mathrm{s}}(r,\varphi) \tag{2}$$

is the sum of the incident and scattered waves, and satisfies the wave equation

$$\triangle u + k^2 V(r) u = 0, \tag{3}$$

where $V(r) = N^2(r)$ and N is the refraction index in Ω. We assume

$$V \equiv 1$$

for the problem in hand (though some generalizations are possible for V depending on r.)

The Dirichlet condition

$$u|_{\partial\Omega} = 0 \tag{4}$$

is valid on the boundary and the traditional Meixner's conditions are implied at the two angular points of the boundary.

In order to formulate the conditions at infinity we introduce the diffracted field

$$u^{\mathrm{d}} := u - u^{\mathrm{go}},$$

where u^{go} is the so-called geometrical optics part of the total field: the sum of incident and possibly reflected from $a_\pm$ waves in the corresponding subdomains of Ω, $r \to \infty$, (see Babich *et al* (2006)). The integral form of the Sommerfeld radiation condition is valid at infinity

$$\int_{-\Phi}^{\Phi} |\partial_r u^{\mathrm{d}} - \mathrm{i}ku^{\mathrm{d}}|^2 R\,\mathrm{d}\varphi \to 0, \qquad R \to \infty. \tag{5}$$

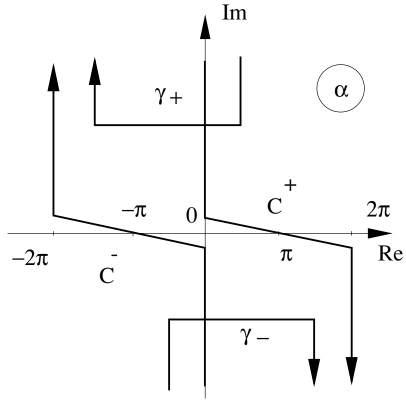

Figure 2. The Sommerfeld double-loop contour and the contours $C^{\pm}$.

The classical solution of the problem in hand is *unique* provided it exists. The corresponding proof is similar to that in Babich *et al* (2006, chapter 1) for the ideal wedge.

2. The generalized Sommerfeld integral and the classical solution of the problem

We look for the solution in the form of the generalized Sommerfeld integral

$$u(r,\varphi) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_+} U(r,\alpha,k)\Sigma(\alpha,\varphi)\,\mathrm{d}\alpha, \tag{6}$$

where γ_+ is the upper loop of the well-known double-loop Sommerfeld contour γ, figure 2 (see also Lyalinov and Zhu (2003, figure 2)).

Contrary to the classical form of the Sommerfeld integral, for which the kernel $U(r,\alpha,k)$ coincides with $\exp\{-\mathrm{i}kr\cos\alpha\}$, the function $U(r,\alpha,k)$ should be chosen appropriately in order to ensure that the integral (6) solves the diffraction problem.

First, we prove a simple statement (which is an extension of the well-known fact for $U(r,\alpha,k) = \exp(-\mathrm{i}kr\cos\alpha)$, $V = 1$; see, e.g., Babich *et al* (2006, chapter 2), Budaev (1995)).

Lemma 1. *Let $U(r,\alpha,k)$ and $\Sigma(\alpha,\varphi)$ be the classical solutions of the equations*

$$\left(r^{-1}\partial_r(r\partial_r) + r^{-2}\partial_\alpha^2 + k^2V(r)\right)U(r,\alpha,k) = 0, \tag{7}$$

$$\left(\partial_\varphi^2 - \partial_\alpha^2\right)\Sigma(\alpha,\varphi) = 0 \tag{8}$$

correspondingly, and

$$\partial_\alpha U(r,\alpha,k)\Sigma(\alpha,\varphi)|_{\partial\gamma_+} = 0, \tag{9}$$

$$U(r,\alpha,k)\partial_\alpha\Sigma(\alpha,\varphi)|_{\partial\gamma_+} = 0, \tag{10}$$

i.e., vanish at the ends of the contour γ_+. Then the integral (6) is a classical solution of the Helmholtz equation (3).

Indeed, we substitute the representation (6) into equation (3), then after simple rearrangements and integration by parts we obtain

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\int_{\gamma_+}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+k^2V(r)\right)U(r,\alpha,k)\Sigma(\alpha,\varphi)\,\mathrm{d}\alpha\\
&\quad=\frac{1}{2\pi \mathrm{i}}\int_{\gamma_+}\left[\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2}+k^2V(r)\right)U(r,\alpha,k)\right\}\Sigma(\alpha,\varphi)\right.\\
&\qquad\left.+\,U(r,\alpha,k)\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\Sigma(\alpha,\varphi)-\Sigma(\alpha,\varphi)\frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2}U(r,\alpha,k)\right]\mathrm{d}\alpha\\
&\quad=\frac{1}{2\pi \mathrm{i}}\int_{\gamma_+}\left[\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2}+k^2V(r)\right)U(r,\alpha,k)\right]\Sigma(\alpha,\varphi)\,\mathrm{d}\alpha\\
&\qquad+\frac{1}{2\pi \mathrm{i}}\int_{\gamma_+}U(r,\alpha,k)\frac{1}{r^2}\left(\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial\alpha^2}\right)\Sigma(\alpha,\varphi)\,\mathrm{d}\alpha=0. \qquad (11)
\end{aligned}$$

Remark. It is assumed that the integral (6) rapidly converges so that its substitution into the Helmholtz equation is justified (see below).

On the second step, we choose the function $\Sigma(\alpha,\varphi)$ in the form

$$\Sigma(\alpha,\varphi)=s(\alpha+\varphi)-s(-\alpha+\varphi) \qquad (12)$$

which obviously satisfies equation (8). Moreover, such a choice is in agreement with the radiation condition at infinity (see Babich *et al* (2006, chapter 2), Budaev (1995)).

Now we consider the boundary conditions on the parts $a_\pm$ of the boundary $\partial\Omega$. They can be ensured by the appropriate choice of the function s. Recalling the problem for the ideal wedge, we take

$$s(\alpha)=\frac{\mu\cos\mu\varphi_0}{\sin\mu\alpha-\sin\mu\varphi_0},\qquad \mu=\pi/2\Phi \qquad (13)$$

and verify that the integral (6) fulfils the boundary condition (4) on $a_\pm$ in view of the equalities

$$s(\alpha\pm\Phi)-s(-\alpha\pm\Phi)=0.$$

Up to now we discussed the choice of the function Σ. The kernel U of the generalized Sommerfeld integral must fulfil equation (7), and, on the other hand, the boundary condition on the arc σ. It is remarkable that the appropriate choice of the kernel for the problem in hand is as follows (Hönl *et al* 1961),

$$U(r,\alpha,k)=U^{\mathrm{i}}(r,\alpha,k)+U^{\mathrm{s}}(r,\alpha,k), \qquad (14)$$

where

$$U^{\mathrm{i}}(r,\alpha,k)=\exp(-\mathrm{i}kr\cos\alpha)=\sum_{m=-\infty}^{\infty}J_m(kr)\,\mathrm{e}^{\mathrm{i}m(\alpha-\pi/2)}, \qquad (15)$$

$$U^{\mathrm{s}}(r,\alpha,k)=-\sum_{m=-\infty}^{\infty}\frac{J_m(ka)}{H_m^{(1)}(ka)}H_m^{(1)}(kr)\,\mathrm{e}^{\mathrm{i}m(\alpha-\pi/2)}. \qquad (16)$$

We observe that formulae (14)–(16) represent the solution of the diffraction problem provided the plane incident wave $U^{\mathrm{i}}(r,\alpha,k)$ is incident[1] on a circle of radius a with the

[1] The direction of incidence is specified by $\alpha_0=0$, provided the incident plane wave has the form (15)

$$U^{\mathrm{i}}(r,\alpha,k)=\exp(-\mathrm{i}kr\cos(\alpha-\alpha_0))|_{\alpha_0=0}.$$

Dirichlet boundary condition. The solution $U^{\mathrm{s}}(r, \alpha, k)$ also satisfies the radiation condition. The solution is obtained for $r \geqslant a$ and $-\pi < \alpha \leqslant \pi$. In order to have the possibility of substituting expressions (14)–(16) into the integral (6) we should continue the solution on the complex plane α. Indeed, the continuation is given by expressions (14)–(16), where $\alpha \in \mathcal{C}$. It is a 2π*-periodic even* function. Moreover, in view of the estimate ($m \to \infty$)

$$\left| \frac{J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(kr) \right| \leqslant \text{const}\, m^{-1/2} \exp\{m \log(ka/2) - m[\log(m) - 1] - m \log(r/a)\}$$

for any fixed $r \geqslant a$, it is an *entire* function of the variable α.

It is obvious from the construction of the kernel in the generalized Sommerfeld integral that expression (6) satisfies, in view of the lemma, the Helmholtz equation and the boundary conditions, provided the integral rapidly converges and can be substituted into the equation and the boundary conditions. So it remains to verify that the sought-for solution is represented by the convergent integral and fulfils the radiation conditions, which shall be discussed in the next sections[2].

It is worth making some comments on the solution obtained. Obviously, the solution in hand is more complex than that for the famous Sommerfeld problem in a non-perturbed angle, figure 1(B). It also depends on the additional parameter a which is the radius of the circular part of the boundary. In particular, this means greater diversity of the wave processes. Let us assume that the parameter ka is large, i.e., consider short-wavelength approximation. In addition to the waves reflected from the angle's sides and from the circular part, one can expect the creeping waves propagating along the circular part of the boundary which attenuate irradiating the energy into the shadow, figure 1(A). This can be demonstrated by means of the appropriate asymptotic analysis not discussed in the present letter.

On the other hand, provided $a \to 0$, the solution in hand reduces to the Sommerfeld solution for the non-perturbed angle. If the parameter ka is of $O(1)$, qualitatively the far field (kr is large) is similar to that in the Sommerfeld problem, figure 2(B), which is not surprising, because for a remote observer the scatterers in figures 1(A) and (B) are indistinguishable. The scattering amplitude of the cylindrical wave from the origin, however, is different as follows from the analysis below and preserves the information on the circular part of the boundary.

3. Convergence of the generalized Sommerfeld integral

In order to study the convergence of integral (6) we should investigate the behaviour of the kernel, or more precisely, of the function

$$\begin{aligned} U^{\mathrm{s}}(r, \alpha, k) &= \frac{\mathrm{i}}{2} \int_{L^+ \cup L^-} \frac{\mathrm{e}^{\mathrm{i}\nu(\pi - \alpha)}}{\sin \pi \nu} A(\nu)\, \mathrm{d}\nu, \\ A(\nu) &= -\mathrm{e}^{-\mathrm{i}\pi\nu/2} \frac{J_\nu(ka)}{H_\nu^{(1)}(ka)} H_\nu^{(1)}(kr), \end{aligned} \tag{17}$$

$\alpha = \alpha_1 + \mathrm{i}\alpha_2$, $\alpha_2 \to \infty$, $\alpha_1 \in (0, \pi)$, where Watson's transformation of the sum (16) has been exploited, the contour is $L^+ = (-\infty + \mathrm{i}\beta, \infty + \mathrm{i}\beta)$ with some small positive β and L^- is the symmetric with respect to the origin contour.

We introduce the large parameter $p = \mathrm{e}^{\alpha_2}$. Exploiting the change of the variable $\nu = p\tau$ in (17) and using the asymptotics of the cylindrical functions, we asymptotically approximate Watson's integral in (17) by the expression

$$U^{\mathrm{s}}(r, \alpha, k) = -\frac{\mathrm{i}p}{2} \int_{L^+ \cup L^-} \frac{\mathrm{e}^{p\psi(\tau, t)}}{\sqrt{2\pi p \tau}} (1 + O_\tau(1/p))\, \mathrm{d}\tau, \tag{18}$$

[2] The Meixner conditions at the corner points can also be followed.

where

$$p\psi(\tau,t) := p[\tau t - p^{-1}\log(\sin\pi p\tau) - \mathrm{i}\pi\tau/2 - \tau(\log(p\tau)-1) + \tau\log((ka)^2/2kr)]$$

and $t = \alpha_2 + \mathrm{i}(\pi - \alpha_1)$. The integral in (18) is written in such a form that the saddle point technique can be directly applied.

The saddle point τ_0 solves the equation

$$p\psi'(\tau_0,t) := p\left[t - \pi\frac{\cos\pi p\tau_0}{\sin\pi p\tau_0} - \mathrm{i}\pi/2 - \log(p\tau_0) + \log((ka)^2/2kr)\right] = 0,$$

then

$$\log(p\tau_0) = t + \mathrm{i}\pi/2 + \log((ka)^2/2kr) + O(\mathrm{e}^{2\pi \mathrm{i}p\tau_0}),$$

provided $\mathrm{Im}\,\tau_0 > 0$ as $\alpha_1 \in (\pi/2,\pi)$, which is assumed. We also obtain

$$\tau_0 = -((ka)^2/2kr)[\sin\alpha_1 + \mathrm{i}\cos\alpha_1](1+o(1)),$$
$$p\psi''(\tau_0,t) = -p/\tau_0 + O(p^2\,\mathrm{e}^{-2\pi p|\tau_0|}),$$
$$p\psi(\tau_0,t) = p\tau_0 + \log 2 + \mathrm{i}\pi/2 + O(p\,\mathrm{e}^{-2\pi p|\tau_0|}).$$

The contour of integration is then deformed in such a manner that its part locally coincides with a segment of the steepest descent path going through τ_0 and the leading asymptotic term is then computed[3] so that the generalized kernel in the Sommerfeld integral (6) satisfies the estimate

$$|U^{\mathrm{s}}(r,\alpha,k)| \leqslant \mathrm{const}|\exp\{-\mathrm{e}^{\alpha_2}[\sin\alpha_1 + \mathrm{i}\cos\alpha_1](ka)^2/(2kr)\}| \tag{19}$$

uniformly with respect to r belonging to any compact domain in $[a,\infty)$, $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, $\alpha_2 \to \infty$, $\alpha_1 \in (0,\pi)$. Exploiting the periodicity and evenness of $U^{\mathrm{s}}(r,\alpha,k)$ and the known properties of $U^{\mathrm{i}}(r,\alpha,k)$, we assert that the integral rapidly converges on the contour γ.

4. The behaviour at infinity

In order to study the behaviour at infinity we represent the solution in the form

$$\begin{aligned} u(r,\varphi) &= \frac{1}{2\pi\mathrm{i}}\int_\gamma U(r,\alpha,k)\mathrm{s}(\alpha+\varphi)\,\mathrm{d}\alpha = \frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\mathrm{s}(\alpha+\varphi)\,\mathrm{d}\alpha + u^{\mathrm{pert}}(r,\varphi), \\ u^{\mathrm{pert}}(r,\varphi) &:= \frac{1}{2\pi\mathrm{i}}\int_\gamma U^{\mathrm{s}}(r,\alpha,k)\mathrm{s}(\alpha+\varphi)\,\mathrm{d}\alpha. \end{aligned} \tag{20}$$

The first summand in (20) is the classical solution for the angle with the Dirichlet boundary conditions, whereas the second term can be considered as the perturbation of the classical Sommerfeld solution due to the circular part σ of the boundary. The classical Sommerfeld integral in (20) obviously satisfies the radiation conditions (5) which is a sum of the incident and non-perturbed reflected from the angle sides waves as well as the non-perturbed cylindrical wave ($r \to \infty$) specified by the saddle points $\pm\pi$ of the integral; see, e.g., Babich *et al* (2006), Buldyrev and Lyalinov (2001).

We turn to the second summand $u^{\mathrm{pert}}(r,\varphi)$ in (20). Deform the double-loop contour γ into $C = C^+ \cup C^-$ shown in figure 2. In the process of such deformation the poles of the integrand (more precisely, of $s(\alpha+\varphi)$) are captured. These poles are

$$\alpha_0 = \varphi_0 - \varphi, \qquad \alpha_{\pm1} = \pm 2\Phi - (\varphi + \varphi_0). \tag{21}$$

[3] It can be shown that the leading term of the asymptotics in the second term of (14) is

$$U^{\mathrm{s}}(r,\alpha,k) \sim -\exp\{-\mathrm{i}(ka)^2/(2kr)\,\mathrm{e}^{-\mathrm{i}\alpha}\}$$

as $\alpha_2 \to \infty$, $\alpha_1 \in (0,\pi)$, $\alpha = \alpha_1 + \mathrm{i}\alpha_2$.

They correspond to the incident and reflected by the angle waves. Addressing to the second summand in (20), we have

$$\begin{aligned} u^{\mathrm{pert}}(r,\varphi) &= \sum_{n=-1,0,+1} H_n U^{\mathrm{s}}(r,\alpha_n,k) + u_0^{\mathrm{pert}}(r,\varphi), \\ u_0^{\mathrm{pert}}(r,\varphi) &= \frac{1}{2\pi\mathrm{i}}\int_C U^{\mathrm{s}}(r,\alpha,k)\mathrm{s}(\alpha+\varphi)\,\mathrm{d}\alpha, \end{aligned} \tag{22}$$

where H_n is zero provided the corresponding pole was not captured, otherwise $H_n = 1$. The sum in (22) can be interpreted as the waves related to the scattering of the incident and reflected on $a_\pm$ plane waves by the circular part σ of the boundary. These terms satisfy the radiation condition at infinity.

Now we reduce the integral $u_0^{\mathrm{pert}}(r,\varphi)$ in (22) to

$$\begin{aligned} u_0^{\mathrm{pert}}(r,\varphi) &= \frac{1}{2\pi\mathrm{i}}\int_C U^{\mathrm{s}}(r,\alpha,k)\mathrm{s}(\alpha+\varphi)\,\mathrm{d}\alpha \\ &= \frac{1}{2\pi\mathrm{i}}\int_{C^+} U^{\mathrm{s}}(r,\alpha,k)[\mathrm{s}(\alpha+\varphi)-\mathrm{s}(-\alpha+\varphi)]\,\mathrm{d}\alpha. \end{aligned}$$

We can change the Hankel function in the kernel U^{s} (see (16)) by its asymptotics

$$H_m^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}}\,\mathrm{e}^{\mathrm{i}(kr-\pi/4-\pi m/2))}(1+O_m(1/kr))$$

and obtain in the leading approximation ($r \to \infty$)

$$\begin{aligned} u_0^{\mathrm{pert}}(r,\varphi) &= \sqrt{\frac{2}{\pi kr}}\,\mathrm{e}^{\mathrm{i}(kr-\pi/4))}\Psi(\varphi,\varphi_0,ka)(1+O(1/kr)), \\ \Psi(\varphi,\varphi_0,ka) &= \frac{1}{2\pi\mathrm{i}}\int_{C^+} A^{\mathrm{s}}(\alpha,ka)[\mathrm{s}(\alpha+\varphi)-\mathrm{s}(-\alpha+\varphi)]\,\mathrm{d}\alpha \end{aligned} \tag{23}$$

with

$$A^{\mathrm{s}}(\alpha,ka) = -\sum_{m=-\infty}^{\infty}\frac{J_m(ka)}{H_m^{(1)}(ka)}\,\mathrm{e}^{\mathrm{i}m(\alpha-\pi)}. \tag{24}$$

The integral in (23) converges on C^+, which is proven by the reduction of (24) to the Watson-type integral and further asymptotic evaluation analogous to that in the previous section.

The scattering amplitude $\Psi(\varphi,\varphi_0,ka)$ describes the influence of the perturbation caused by the circular arc σ to the far field. This arc plays the role of a virtual source at the angle vertex producing the far field with the scattering amplitude $\Psi(\varphi,\varphi_0,ka)$.

5. Conclusion

Considering a simple example of the diffraction problem, we demonstrated that the generalized form of the Sommerfeld integral can be applied to a broader class of diffraction problems than it was implied before. We also emphasize that the generalized form of the integral could be exploited for the other problems like diffraction by an angle with the circular material coating centred at the vertex or by a dielectric sphere with the centre at the vertex of the perfectly conducting cone, etc (see also Tai (1994), Luk'yanov and Nikitin (2000), Lavrov and Luk'yanov (2002)). It seems that extension of the approach to the case of the impedance-type boundary conditions on the infinite part of the boundary, if exists, is non-trivial.

Acknowledgments

The author is thankful to Dr.-Eng. Ning Yan Zhu from Stuttgart University for pointing the work Tai (1994) and for the fruitful discussions.

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